

Orbit-equivalent infinite permutation groups

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Abstract

Let G, H be closed permutation groups on an infinite set X , with H a subgroup of G . It is shown that if G and H are *orbit-equivalent*, that is, have the same orbits on the collection of finite subsets of X , and G is primitive but not 2-transitive, then $G = H$.

Keywords: primitive permutation group, orbit-equivalent, set-homogeneous.

1 Introduction

We consider closed permutation groups acting on an infinite set X ; that is, subgroups of $\text{Sym}(X)$ which are closed in $\text{Sym}(X)$ in the topology of *pointwise convergence* on $\text{Sym}(X)$ with respect to the discrete topology on X (so the basic open sets are cosets of pointwise stabilisers of finite sets). It is easily checked that a closed permutation group on X is precisely the automorphism group of a relational structure with domain X . Two permutation groups G, H on the set X are said to be *orbit-equivalent* if, for every positive integer k , G and H have the same orbits on the collection of unordered k -element subsets of X , denoted here by $X^{[k]}$. This generalises a definition for finite permutation groups. Observe that if G, H are orbit-equivalent, then they are each orbit-equivalent to $\langle G, H \rangle$. Thus, to investigate such pairs, it suffices to consider G, H with H a subgroup of G . Easily, if $H \leq G$ and G, H are orbit-equivalent, then G is transitive (on X) if and only if H is transitive, and also G and H preserve the same

systems of imprimitivity on X ; so G is primitive on X (that is, preserves no proper non-trivial equivalence relation on X) if and only if H is primitive.

Our main theorem is the following. Our particular interest is in the case when X is countably infinite, but the proofs below do not use countability.

Theorem 1.1. *Let G, H be orbit-equivalent closed permutation groups on the infinite set X , with $H \leq G$, and suppose that G is primitive but not 2-transitive. Then $H = G$.*

We stress that if H is a closed proper subgroup of $G \leq \text{Sym}(X)$, then for some $k > 0$, some G -orbit on X^k (the set of k -tuples from X) breaks into more than one H -orbit. The assumption in the theorem that G and H are closed seems essential; indeed, any subgroup H of $\text{Sym}(X)$ is orbit-equivalent to its closure, and, for example, the dense (and so orbit-equivalent) subgroups of $\text{Sym}(X)$ are exactly the subgroups of $\text{Sym}(X)$ which are k -transitive for all positive integers k , and these seem hopelessly unclassifiable.

This paper takes its motivation from two sources. First, there is an extended literature on primitive orbit-equivalent pairs of permutation groups on a *finite* set X ; see for example [20, 11, 21]. Clearly, the symmetric and alternating groups Sym_n and Alt_n , in their natural actions on $\{1, \dots, n\}$, are orbit-equivalent for $n \geq 3$. Also, if G is a permutation group on a finite set X and has a regular orbit U on the power set $\mathcal{P}(X)$, and H is a proper subgroup of G , then H is intransitive on U , and so H is not orbit-equivalent to G . It is shown in [3] that if X is finite then there are just finitely many primitive subgroups of $\text{Sym}(X)$ which do not contain $\text{Alt}(X)$ and have no regular orbit on $\mathcal{P}(X)$ (and so *could* have an orbit-equivalent proper subgroup). Such primitive groups G (with no regular orbit on X) are classified by Seress in [18], who then classifies all pairs of finite primitive orbit-equivalent permutation groups (H, G) with $H < G$. There is further work on the finite imprimitive case in [19].

The second source of motivation is more model-theoretic, namely the study of homogeneous structures. Recall that a countable (possibly finite) structure M in a first order relational language is said to be *homogeneous* if every isomorphism between finite substructures of M extends to an automorphism of M . A natural generalisation, originally considered by Fraïssé in [8], is to say that the countable structure M is *set-homogeneous* if, whenever U, V are isomorphic finite substructures of M , there is $g \in \text{Aut}(M)$ with $U^g = V$. Finite set-homogeneous graphs are classified by Ronsse in [17], and a very short proof was given by Enomoto in [7] that every finite set-homogeneous graph is homogeneous. There is a classification of set-homogeneous digraphs (allowing two vertices to be linked by an arc in each direction) in [9], building on a corresponding classification of finite homogeneous digraphs by Lachlan [12]. Also, there are initial results on countably infinite set-homogeneous structures, in particular graphs and digraphs, in [6] and [9]. The latter paper poses the following related question: given a homogeneous structure M , when does $\text{Aut}(M)$ have a proper closed subgroup H which acts set-homogeneously on M , that is, has the same orbits as $\text{Aut}(M)$ on the collection of unordered finite subsets of M ? Equivalently, for which M does $\text{Aut}(M)$ have a proper closed orbit-equivalent subgroup? (Here, and throughout the paper, we use the same symbol M for a structure and for its domain.)

A countably infinite set X in the empty language is homogeneous, and has automorphism group $\text{Sym}(X)$. By a theorem of Cameron [2], $\text{Sym}(X)$ has just four orbit-equivalent closed proper subgroups, namely $\text{Aut}(X, <)$, $\text{Aut}(X, B)$, $\text{Aut}(X, C)$, and $\text{Aut}(X, S)$. Here $<$ is a dense linear order without end points on X , B is the (ternary) linear betweenness relation on X induced from $<$, C is the (also ternary) circular order on X induced from $<$, and S is the corresponding arity 4 separation relation. Observe that $\text{Aut}(X, S) = \langle \text{Aut}(X, B), \text{Aut}(X, C) \rangle$ and is 3-transitive but not 4-transitive. Our conjecture below would strengthen Theorem 1.1 by removing the ‘not 2-transitive’ assumption.

Conjecture 1.2. *Let G and H be distinct orbit-equivalent primitive closed permutation groups on a countably infinite set X . Then G and H belong to the list $\text{Aut}(X, <)$, $\text{Aut}(X, B)$, $\text{Aut}(X, C)$, $\text{Aut}(X, S)$, $\text{Sym}(X)$ described above.*

Recall the following standard terminology, for a permutation group G on a set X , and an integer $k > 0$: G is *k-transitive* if it is transitive on the ordered k -subsets of X ; and G is *k-homogeneous* if it is transitive on the unordered k -subsets of X . Also, if U is a subset of X then $G_{\{U\}}$ and $G_{(U)}$ denote respectively the setwise and pointwise stabilisers of U in G , and if $x \in X$ then $G_x := \{g \in G : x^g = x\}$.

The proof of Theorem 1.1 splits into two cases:

- (1) G is primitive but not 2-homogeneous;
- (2) G is 2-homogeneous (and so primitive) but is not 2-transitive.

Our main tool for both cases is the notion of *local rigidity*. We shall say that a permutation group G acting on an infinite set X acts *locally rigidly* if for all finite $U \subset X$, there is some finite $V \subset X$ such that $U \subseteq V$ and the setwise stabiliser $G_{\{V\}}$ of V fixes U pointwise. Likewise, a first order relational structure M is *locally rigid* if, for every finite substructure U of M , there is a finite substructure V of M containing U such that every automorphism of V fixes U pointwise. Clearly, if a relational structure M is locally rigid, then any subgroup of its automorphism group acts locally rigidly on M . Strengthening the notion of local rigidity, we shall later say that a countably infinite first order structure M is *cofinally rigid* if, for every finite substructure U of M , there is a finite substructure V of M with $U \subseteq V$ such that the automorphism group of V is trivial. Here, ‘substructure’ is used in the standard model-theoretic sense, corresponding to the graph-theoretic notion of ‘induced subgraph’.

Lemma 1.3. *Let G, H be closed permutation groups on X , with $H \leq G$. If G and H are orbit-equivalent and G acts locally rigidly, then $H = G$.*

Proof. It suffices to show that H has the same orbits as G on X^k for all k . So let $\bar{u}_1, \bar{u}_2 \in X^k$ be in the same orbit of G ; that is, there is $g \in G$ such that $\bar{u}_1^g = \bar{u}_2$. Let $U_1, U_2 \subset X$ be enumerated by \bar{u}_1, \bar{u}_2 respectively. Since G acts locally rigidly on X , there is finite $V_1 \subset X$ such that $U_1 \subseteq V_1$ and $G_{\{V_1\}} \leq G_{(U_1)}$. Let $V_2 := V_1^g$. Then V_1, V_2 are in the same orbit of G , so by orbit-equivalence there is some $h \in H$ such that $V_1^h = V_2$. Now $gh^{-1} \in G_{\{V_1\}}$, so in fact $gh^{-1} \in G_{(U_1)}$. Thus $\bar{u}_1^h = \bar{u}_2$ as required. \square

In both cases (1) and (2) (G primitive, and either not 2-homogeneous, or 2-homogeneous but not 2-transitive) we shall show that G acts locally rigidly on X . In fact, in the second case we show that G is a group of automorphisms of a *cofinally rigid* tournament. Our method to show the local rigidity of such actions stems from a similar result in [6], which we adapt. Formally, we view a graph Γ as a relational structure $\Gamma = (X, R)$, where R is a symmetric irreflexive binary relation on X . Given a graph Γ , if x, y are vertices we write $x \sim y$ if x and y are adjacent, and let $\Gamma(x) := \{v \in X : v \sim x\}$, the *neighbour set* of x . We shall prove in Lemma 2.3 a strengthening of the following result.

Lemma 1.4. [6] *Let Γ be an infinite graph such that, for all distinct vertices x, y of Γ , the sets $\Gamma(x) \setminus \Gamma(y)$ and $\Gamma(y) \setminus \Gamma(x)$ are both infinite. Then Γ is locally rigid.*

We draw attention to a basic Ramsey-theoretic principle which is well-known, for example in model theory, and used below in both the primitive not 2-homogeneous case, and the 2-homogeneous not 2-transitive case.

Definition 1.5. Let L be a finite relational language, let M be a first order L -structure, A a finite subset of the domain of M , and P_1, \dots, P_r disjoint subsets of $M \setminus A$, with $P_i := \{p_{i,0}, \dots, p_{i,n-1}\}$ for each $i = 1, \dots, r$. We say that P_1, \dots, P_r are *mutually indiscernible over A* if the following holds for any positive integers $e_1, \dots, e_r < n$: for each $j = 1, \dots, r$, let \bar{p}_j, \bar{p}'_j be e_j -tuples from P_j , each listed in increasing order (so if $\bar{p}_j = (p_{j,i(1)}, \dots, p_{j,i(e_j)})$, then $i(1) < \dots < i(e_j)$); then the map taking \bar{p}_j to \bar{p}'_j for each j , extended by the identity on A , is an isomorphism of L -structures.

Lemma 1.6. *Let M, L, A be as in Definition 1.5 with M infinite, and let Q_1, \dots, Q_r be countably infinite disjoint subsets of $M \setminus A$. Let n be a positive integer. Then the following hold.*

- (i) *There are subsets $P_1 \subset Q_1, \dots, P_r \subset Q_r$, each of size n , such that P_1, \dots, P_r are mutually indiscernible over A (with respect to some indexing of each P_i).*
- (ii) *If every relation of L is of arity at most 2, and P_1, \dots, P_r are as in (i), then for each $i = 1, \dots, r$, either some relation of L induces a total order on P_i , or every permutation of P_i , extended by the identity on $S_i := A \cup \bigcup_{j \neq i} P_j$, is an automorphism of the induced L -structure on $S := A \cup P_1 \cup \dots \cup P_r$.*

Proof. (Sketch) (i) Let $Q_i := \{q_{i,j} : j \in \mathbb{N}\}$ for each $i = 1, \dots, r$. Let d be the maximum arity of a relation in L . Colour each subset $\{i_1, \dots, i_d\}$ of \mathbb{N} in such a way that given natural numbers $i_1 < \dots < i_d$ and $k_1 < \dots < k_d$, the map

$$(q_{1,i_1}, \dots, q_{1,i_d}, \dots, q_{r,i_1}, \dots, q_{r,i_d}) \mapsto (q_{1,k_1}, \dots, q_{1,k_d}, \dots, q_{r,k_1}, \dots, q_{r,k_d})$$

is an isomorphism over A if and only if $\{i_1, \dots, i_d\}$ and $\{k_1, \dots, k_d\}$ have the same colour. By Ramsey's Theorem, replacing \mathbb{N} by an infinite monochromatic subset if necessary, we may suppose that \mathbb{N} is monochromatic. Now let $p_{i,j} := q_{i,(i-1)n+j}$ for each $i = 1, \dots, r$ and $j = 0, \dots, n-1$. Put $P_i := \{p_{i,1}, \dots, p_{i,n-1}\}$ for each $i = 1, \dots, r$. Then P_1, \dots, P_r are mutually indiscernible over A .

- (ii) This is immediate from (i).

□

The case of Theorem 1.1 when G is primitive but not 2-homogeneous is handled in Section 2, and the 2-homogeneous but not 2-transitive case is treated in Section 3. Section 4 consists of some further observations, about bounds in local rigidity, approaches to Conjecture 1.2, and regular orbits on the power set. We also observe that our proofs give a slight strengthening of Theorem 1.1, namely Theorem 4.1.

2 G primitive but not 2-homogeneous

In this section we prove the following.

Proposition 2.1. *Let G be a primitive but not 2-homogeneous permutation group on an infinite set X . Then the action of G on X is locally rigid.*

The proposition follows rapidly from the following two lemmas. The first uses an argument in [15, Proposition 4.4].

Lemma 2.2. *Let G be a primitive but not 2-homogeneous permutation group on an infinite set X . Then there is a G -invariant graph Γ with vertex set X such that for all distinct $x, y \in X$, the symmetric difference $\Gamma(x) \Delta \Gamma(y)$ is infinite.*

Proof. Let U be any G -orbit on the collection of 2-subsets of X . Then U is the edge set of a G -invariant graph Γ_0 with vertex set X , and as G is not 2-homogeneous, Γ_0 is not complete. For $x \in X$, write $\Gamma_0(x)$ for the neighbour set of x in Γ_0 . Define the equivalence relation \equiv_0 on X , putting $x \equiv_0 y$ if and only if $|\Gamma_0(x) \Delta \Gamma_0(y)|$ is finite. Then \equiv_0 is G -invariant, so by primitivity \equiv_0 is trivial or universal. The lemma holds if \equiv_0 is trivial, so we shall suppose that \equiv_0 is universal.

Recall that a graph is *locally finite* if all of its vertices have finite degree.

Claim. Either Γ_0 or its complement is locally finite.

Proof of Claim. Suppose not, and fix $x \in X$. Then both $\Gamma_0(x)$ and $X \setminus \Gamma_0(x)$ are infinite. If $y \in \Gamma_0(x)$ then (as \equiv_0 is universal) $\Gamma_0(y) \setminus \Gamma_0(x)$ is finite. Hence as G_x has at most two orbits on $\Gamma_0(x)$ there is $k \in \mathbb{N}$ such that for all $y \in \Gamma_0(x)$, we have $|\Gamma_0(y) \setminus \Gamma_0(x)| \leq k$. Pick distinct $z_1, \dots, z_{k+1} \in X \setminus (\{x\} \cup \Gamma_0(x))$. Then as $x \equiv_0 z_i$ for each i , each set $\Gamma_0(z_i) \cap \Gamma_0(x)$ is cofinite in $\Gamma_0(x)$. Hence there is $y \in \Gamma_0(x) \cap \bigcap_{i=1}^{k+1} \Gamma_0(z_i)$. Then $z_1, \dots, z_{k+1} \in \Gamma_0(y) \setminus \Gamma_0(x)$, so $|\Gamma_0(y) \setminus \Gamma_0(x)| \geq k+1$, which is a contradiction.

By the claim, replacing Γ_0 by its complement if necessary, we may suppose that Γ_0 is locally finite. By our original assumption that Γ_0 is not complete (or null), Γ_0 has an edge. By primitivity, Γ_0 is connected. Now let Γ be the graph on X whose edge set consists of the set of unordered pairs an even distance apart in Γ_0 . Then Γ is also G -invariant. Pick $v_0 \in X$, and choose a Γ_0 -path $v_0 \sim v_1 \sim v_2 \sim \dots$ so that the distance $d_0(v_0, v_i)$ between v_0 and v_i in Γ_0 equals i for each i (this is certainly possible, for example by König's Lemma). Then $v_{2i} \in \Gamma(v_0) \setminus \Gamma(v_1)$ for each $i > 0$. Thus $\Gamma(v_0)$ and $\Gamma(v_1)$ have infinite symmetric difference, and since G is primitive, this holds for all pairs of distinct vertices in Γ . \square

In the next lemma, and later in the paper, if A, B are sets we write $A \subset_f B$ if $B \setminus A$ is infinite and $A \setminus B$ is finite. The lemma below extends Lemma 1.4, since under the assumptions of that lemma, $x < y$ (as defined below) never holds. If u, v, w are distinct vertices of the graph Γ , we say w *separates* u and v if $w \in \Gamma(u) \Delta \Gamma(v) \setminus \{u, v\}$, and call a collection of such vertices w a *separating set* for u and v .

Lemma 2.3. *Let $\Gamma = (X, R)$ be an infinite graph, and suppose that $\Gamma(x) \Delta \Gamma(y)$ is infinite for any distinct $x, y \in X$. Write $x < y$ whenever $\Gamma(x) \supset_f \Gamma(y)$. Then the structure $\Gamma_{<} = (X, R, <)$ is locally rigid.*

Proof. We slightly adapt the proof of Proposition 6.1 from [6]. So let $U = \{u_1, \dots, u_n\}$ be a finite subset of X . We aim to find finite V with $U \subset V \subset X$, such that $\text{Aut}(V, R, <)$ fixes U pointwise.

For each $u_i, u_j \in U$, with $i < j$, we find an infinite separating set $Q_{ij} \subset X \setminus U$ as follows: if $u_i < u_j$, then let $Q_{ij} \subset \Gamma(u_i) \setminus (\Gamma(u_j) \cup \{u_j\})$; if $u_j < u_i$, then let $Q_{ij} \subset \Gamma(u_j) \setminus (\Gamma(u_i) \cup \{u_i\})$; and if $u_i \parallel u_j$ (that is, u_i, u_j are incomparable under $<$), then let $Q_{ij} \subset \Gamma(u_i) \setminus (\Gamma(u_j) \cup \{u_j\})$.

Let K be a positive integer. By Lemma 1.6 with respect to the language $L = \{R, <\}$, we can choose for each $i < j$ a subset P_{ij} of Q_{ij} with $|P_{ij}| = K$, such that the collection of all sets P_{ij} is mutually indiscernible over U . Let $W = U \cup \bigcup (P_{ij} : 1 \leq i < j \leq n)$. Then each P_{ij} carries a complete or null induced graph structure, and for each $x, y \in P_{ij}$ and $z \in W \setminus P_{ij}$, we have $x \sim z$ if and only if $y \sim z$.

For any subset Y of X , define the equivalence relation \approx_Y on Y , where, for $x, y \in Y$, $x \approx_Y y$ if and only if $(\Gamma(x) \Delta \Gamma(y)) \cap Y \subseteq \{x, y\}$. Then \approx_Y -classes always carry a complete or null (that is, independent set) induced subgraph structure. If Z is an \approx_Y -class, then for $z_1, z_2 \in Z$ and $y \in Y \setminus Z$, we have $y \sim z_1 \Leftrightarrow y \sim z_2$; in particular, $\text{Aut}(Y)_{(Y \setminus Z)}$ induces $\text{Sym}(Z)$. Observe that if $Y_1 \subset Y_2 \subseteq X$ and $x, y \in Y_1$, then $x \approx_{Y_2} y$ implies $x \approx_{Y_1} y$. We often identify such Y with the induced subgraph $(Y, R \cap Y^2)$ of Γ which it carries. Thus, \approx_Y is $\text{Aut}(Y, R)$ -invariant.

Now, each P_{ij} lies in a \approx_W -class of W . Deleting some sets P_{ij} if necessary (only where elements of distinct sets P_{ij} are \approx_W -equivalent, and retaining the assumption that any two distinct elements of U are separated by some set of form P_{ij}), we may suppose: no two elements x, y in distinct sets P_{ij}, P_{kl} are \approx_W -equivalent. Also, \approx_W -classes contain at most one point of U ; for if $u_i, u_j \in U$ with $i < j$ then there is a non-empty set P_{kl} whose elements separate u_i and u_j , so witness that $u_i \not\approx_W u_j$. Let $m = \binom{n}{2}$, an upper bound on the number of distinct sets P_{ij} . Adjusting the P_{ij} and hence W further, we arrange the sizes of the P_{ij} so that $|P_{ij}| \geq 2$ for each i, j and distinct \approx_W -classes of W of size at least two all have different sizes, with size at most $m + 1 \in \mathbb{N}$. Now every \approx_W -class of W of size greater than 1 consists of a set P_{ij} , possibly together with an element of U . We will say that a set $Y \subseteq X$ is *huge* if $|Y| > m + 1$.

Any automorphism of (W, R) will preserve \approx_W , and will fix setwise each \approx_W -class of size at least two (as these classes all have different sizes). Hence, if no element of U is \approx_W -equivalent to any element of any P_{ij} (that is, elements of U lie in \approx_W -classes of size 1), then as the P_{ij} separate the elements of U , any automorphism of W will fix U pointwise, as required. So the concern is that some \approx_W -class C in W of size

at least two might consist of a set P_{ij} together with some $u \in U$, in which case there would be an automorphism of (W, R) mapping u to some vertex in $C \setminus \{u\}$.

So suppose $u \in C \cap U$ as in the last paragraph. By the Pigeonhole Principle (retaining all the above reductions, so initially working with larger sets P_{ij}) we may suppose for all such C, u that either $u \parallel c$ (that is, u and c are incomparable with respect to $<$) for all $c \in C \setminus \{u\}$, or $u < c$ for all $c \in C \setminus \{u\}$, or $c < u$ for all $c \in C \setminus \{u\}$. For such u, c and C , we add a finite set S_{cu} of additional vertices of Γ to W according to the following recipe.

If C is null, then for each $c \in C \setminus \{u\}$ for which $c \not\asymp u$, the set $\Gamma(c) \setminus \Gamma(u)$ is infinite, and we choose $S_{cu} \subset \Gamma(c) \setminus (\Gamma(u) \cup W)$. If C is complete, then for each $c \in C \setminus \{u\}$ for which $c \not\asymp u$, the set $\Gamma(u) \setminus \Gamma(c)$ is infinite, and we choose $S_{cu} \subset \Gamma(u) \setminus (\Gamma(c) \cup W)$. In other cases (C null and $c > u$ for all $c \in C \setminus \{u\}$, or C complete and $c < u$ for all $c \in C \setminus \{u\}$) we do not add any corresponding set S_{cu} . Each S_{cu} (for $u \in U$ and $c \in W \setminus U$ with $c \approx_W u$) is chosen to be huge, and these sets are chosen so that if $(c, u) \neq (c', u')$ then $S_{cu} \cap S_{c'u'} = \emptyset$. We may suppose, again by the Pigeonhole Principle, that for each such c, u , either $c \parallel x$ for all $x \in S_{cu}$, or $c < x$ for all $x \in S_{cu}$, or $x < c$ for all $x \in S_{cu}$. By Lemma 1.6 with respect to $L = \{R, <\}$, we may suppose that the collection of all such sets S_{cu} is mutually indiscernible over W (formally, before applying the lemma, the S_{cu} may be taken to be infinite). Let V be the union of all such sets S_{cu} and of W . Observe that each S_{cu} is either complete or null, and for each $x, y \in S_{cu}$ and $z \in V \setminus S_{cu}$, we have $x \sim z$ if and only if $y \sim z$. In particular, any two elements of a set S_{cu} are \approx_V -equivalent. We arrange that all elements of $V \setminus W$ lie in huge \approx_V -classes, and that distinct huge \approx_V -classes have different sizes, so each is fixed setwise by any automorphism of (V, R) .

We aim to show that every automorphism of $(V, R, <)$ must fix U pointwise, which will complete the proof of the lemma. As a first step, observe that every huge \approx_V -class S contains some set S_{cu} . We claim that no huge \approx_V -class meets U . For suppose S is a huge \approx_V -class, with $a \in U \cap S$. There is $u \in U$ and $c \in W \setminus U$, and a \approx_W -class C with $c, u \in C$, such that $S \supset S_{cu}$. Clearly $a = u$, since otherwise, as S_{cu} separates c from u , a would separate c and u and lie in $U \subset W$, contradicting that $c \approx_W u$. Now if C is null then, by our rule for the process adding S_{cu} , all vertices of $S \setminus \{u\}$ are adjacent to c ; hence c separates u from other elements of S , so $u \notin S$, a contradiction. Likewise, if C is complete, then all vertices of $S \setminus \{u\}$ are non-adjacent to c , so again c separates u from the rest of S , so $u \notin S$. This proves the claim.

Claim. Let $g \in \text{Aut}(V, R, <)$. Then there is $h \in \text{Aut}(V, R, <)_{(U)}$ such that gh fixes W setwise.

Proof of Claim. There are distinct (so different-sized) huge \approx_V -classes S_j (for $j \in J$), each fixed setwise by g , such that $V \setminus W \subseteq \bigcup_{j \in J} S_j$. We may assume that W is not fixed setwise by g , as otherwise the claim is trivial. Hence, for some $j \in J$, we have $(S_j \cap (V \setminus W))^g \neq S_j \cap (V \setminus W)$.

First, we show that $|S_j \cap W| = 1$. There are $u \in U$, and some \approx_W -class C containing distinct elements u, c of W , such that $S_j \supseteq S_{cu}$. We may suppose that C is null, and $S_{cu} \subset \Gamma(c) \setminus \Gamma(u)$, as the other case where C is complete and $S_{cu} \subset \Gamma(u) \setminus \Gamma(c)$ is similar. Now no element of $W \setminus \{u, c\}$ could lie in S_j , for otherwise it would separate

u from c in W contradicting that $u \approx_W c$. Hence, $S_j \cap W \subseteq \{c\}$, so due to the existence of the element g , we have $S_j \cap W = \{c\}$. In fact, $S_j = S_{cu} \cup \{c\}$: for if $c' \in C \setminus \{u, c\}$ then $S_{c'u} \neq \emptyset$ but c' separates elements of $S_{c'u}$ from c so elements of $S_{c'u}$ do not lie in S_j ; and if $u' \in U \setminus \{u\}$, then no set of form $S_{du'}$ could be a subset of S_j , for otherwise c (in W) would separate d from u' so the set $S_{du'}$ would not have been added.

By our assumption, there is $v \in S_{cu}$ such that $v^g = c$. It is not possible that S_{cu} is totally ordered by $<$; this follows easily from the facts that g induces an automorphism of $(S_{cu} \cup \{c\}, <)$, and the earlier assumption that either $c < x$ for all $x \in S_{cu}$, or $x < c$ for all $x \in S_{cu}$, or $c \parallel x$ for all $x \in S_{cu}$. It follows by Lemma 1.6(ii) that any permutation of S_{cu} , extended by the identity on the rest of V , is an automorphism of $(V, R, <)$. In particular distinct elements of S_{cu} are $<$ -incomparable, so as $v^g = c$, S_j is an antichain with respect to $<$. Now it could not happen that there is some $t \in V \setminus S_j$ whose $<$ -relation to c is different from its $<$ -relation to all other elements of S_j . For otherwise $t^{g^{-1}}$ would have a different $<$ -relation to v and to all other elements of S_j , contradicting the mutual indiscernibility in the construction of S_{cu} . It follows that if g' is the inverse of g on S_j and the identity on the rest of V , then $g' \in \text{Aut}(V, R, <)$. The element h of the claim will be a product of elements of the form g' , each acting on a different huge \approx_V -class.

To finish the proof of the lemma, let $g \in \text{Aut}(V, R, <)$, and let h be as in the claim. We must show $u^g = u$ for all $u \in U$. Now by construction gh fixes W setwise, and we claim that gh fixes U setwise. Indeed, suppose for a contradiction that $u \in U$ and $u^{gh} \notin U$. As the \approx_W -classes of W of size greater than one are all of different sizes, they are all fixed setwise by gh . Hence, as all elements of $W \setminus U$ lie in \approx_W -classes of size greater than one, u^{gh} and hence also u lie in some \approx_W -class C of size greater than one. Now, by the construction of V from U , either u is the greatest or least element of C with respect to $<$, or u and u^{gh} are separated by some huge set of form $S_{u, u^{gh}}$. The first case is impossible as gh preserves $<$. The second case is also impossible, since as the huge \approx_V -classes all have different sizes, they are fixed setwise by gh . Thus, as claimed, gh induces an automorphism of $(W, R, <)$ which fixes U setwise. Hence gh fixes U pointwise; for any two distinct elements of U are separated by an \approx_W -class of size greater than one, and all such classes have different sizes, so are fixed setwise by gh . Thus, g fixes U pointwise. \square

Remark 2.4. Careful inspection of the above proof shows that if $|U| = n$, then V may be chosen to have size at most $O(n^8)$. For in constructing W from U , if $m = \binom{n}{2}$ we add at most m sets P_{ij} , each of size at least 2 and all of different sizes, so $|W| = n + k$ where $k := |W \setminus U| \leq \frac{(m+1)(m+2)}{2} - 1$. Then in adding the sets S_{cu} to obtain V , we add at most k such sets, each of size at least $m + 2$, and all of different sizes. Thus, $|V \setminus W| \leq (m + 2) + (m + 3) + \dots + (m + k + 1) = \frac{k}{2}(2m + k + 3)$. Thus, $|V| \leq \frac{1}{2}(2n + k(2m + k + 5))$. This is used in Theorem 4.1 below.

Proof of Proposition 2.1. By Lemma 2.2, there is a G -invariant graph Γ on X such that for all distinct $x, y \in X$, the set $\Gamma(x) \Delta \Gamma(y)$ is infinite. The partial order $<$ defined in Lemma 2.3 is clearly also G -invariant. The proposition thus follows immediately from that lemma. \square

3 G 2-homogeneous but not 2-transitive

By Proposition 2.1, to complete the proof of Theorem 1.1 it suffices to prove the following.

Proposition 3.1. *Let G be a 2-homogeneous but not 2-transitive permutation group on an infinite set X . Then the action of G on X is locally rigid.*

Recall that a *tournament* is a directed loopless digraph (T, \rightarrow) such that for any distinct vertices x, y , exactly one of $x \rightarrow y$ or $y \rightarrow x$ holds. A group which is 2-homogeneous but not 2-transitive has just one orbit on unordered 2-sets, but two orbits on ordered pairs of distinct elements. Each of these orbits is the arc set of a G -invariant tournament with vertex set X . Thus, to prove Proposition 3.1, we develop analogues of the methods of Section 2, but for tournaments.

Let \rightarrow denote the arc relation in a tournament $T = (X, \rightarrow)$, and let $G = \text{Aut}(T)$. For $x \in X$, we let $\Gamma^+(x) := \{y \in X : x \rightarrow y\}$, the set of *outneighbours* of x . For $x, y, z \in X$, we say that z *separates* x, y if $x \rightarrow z \rightarrow y$ or $y \rightarrow z \rightarrow x$. Furthermore $Z \subset X$ *separates* x, y if each $z \in Z$ separates x, y . We write $x \rightarrow Z$ if $x \rightarrow z$ for each $z \in Z$.

Proposition 3.2. *Let $T = (X, \rightarrow)$ be an infinite tournament such that for any distinct $x, y \in X$, the sets $\Gamma^+(x) \setminus \Gamma^+(y)$ and $\Gamma^+(y) \setminus \Gamma^+(x)$ are both infinite. Then T is cofinally rigid.*

We first isolate an easy lemma, used to prove Proposition 3.2, in case it has other uses. It may be known.

Let $T = (X, \rightarrow)$ be a tournament. We will say that $A \subset X$ is a *nice* set if $A \neq \emptyset$ and for all $a_1, a_2 \in A$ and $v \in X \setminus A$, we have $a_1 \rightarrow v$ if and only if $a_2 \rightarrow v$. (That is, all vertices in a nice set are related in the same way to vertices outside the nice set; equivalently, no vertex outside a nice set separates a pair of vertices inside the nice set.) Note that vacuously any singleton is a nice set, and X is nice. Furthermore, we will say that $A \subset X$ is a *good* set, if A is totally ordered by \rightarrow and is nice. We consider the *maximal* good subsets of X , that is, good sets A such that there is no good set $A' \subset X$ with $A' \supset A$.

Lemma 3.3. *If $T = (X, \rightarrow)$ is a tournament, then the maximal good subsets of X form a partition of X .*

Proof. We claim that if A is good and $B \neq A$ is maximal good (where $A, B \subseteq X$), then either $A \subset B$ or $A \cap B = \emptyset$. To see this, let $d \in A \cap B$, and let $C = A \cup B$. We show that C is good, which ensures $B = C$.

Let $c_1, c_2 \in C$, $v \in X \setminus C$. Now $c_1 \rightarrow v$ if and only if $d \rightarrow v$ if and only if $c_2 \rightarrow v$. This holds because A and B are both nice and $d \in A \cap B$. Hence C is nice. If C is not totally ordered, then there is some 3-cycle $c_1 \rightarrow c_2 \rightarrow c_3 \rightarrow c_1$ in C . Since A and B are both totally ordered, we must have at least one of these points in $A \setminus B$ and one in $B \setminus A$. Suppose $c_1 \in A \setminus B$ and $c_2 \in B \setminus A$ (the other case is similar). Then if $c_3 \in A$, then c_2 separates c_1, c_3 , contradicting the fact that A is nice. Otherwise

$c_3 \in B$, then similarly c_1 separates c_2, c_3 , contradicting the fact that B is nice. Hence C is totally ordered. Now $B \subseteq C$, and C is good, so $A \subseteq B = C$ by maximality of B .

The lemma follows immediately from the claim (using Zorn's Lemma if X is infinite), since each singleton in X is a good set. \square

Proof of Proposition 3.2. Let $U = \{u_1, \dots, u_n\} \subset X$. For any distinct $i, j \in \{1, \dots, n\}$, the set $\Gamma^+(u_i) \setminus \Gamma^+(u_j) = \{v \in X : u_i \rightarrow v \rightarrow u_j\}$ is infinite. Hence by Ramsey's Theorem, there is $U_{ij} \subseteq \Gamma^+(u_i) \setminus (\Gamma^+(u_j) \cup \{u_j\})$ with $|U_{ij}| = \aleph_0$, such that U_{ij} is totally ordered by \rightarrow . Note that the sets U_{ij}, U_{ji} both separate u_i, u_j (since $u_i \rightarrow U_{ij} \rightarrow u_j$, and $u_j \rightarrow U_{ji} \rightarrow u_i$). We may choose the U_{ij} so that if $(i, j) \neq (k, l)$ then $U_{ij} \cap U_{kl} = \emptyset$.

Claim 1. Let N be any positive integer. Then there are finite subsets V_{ij} of U_{ij} (for all distinct integers i, j with $1 \leq i, j \leq n$) of size N such that the following holds, where T' is the induced subtournament of T with vertex set $U \cup \bigcup_{i \neq j} V_{ij}$: for any distinct $i, j \in \{1, \dots, n\}$, and for each $x, y \in U_{ij}$ and $v \in T' \setminus U_{ij}$, $x \rightarrow v$ if and only if $y \rightarrow v$.

Proof of Claim 1. This is an immediate application of Lemma 1.6.

Provided we initially choose N large enough, we may cut the V_{ij} down further, and so suppose that each set V_{ij} has size exactly 2^r for some $r \geq 2$, and that distinct sets V_{ij} and V_{kl} have distinct sizes. Observe (for use in Theorem 4.1) that T' has $n + \sum_{i=2}^{m+1} 2^i$ vertices where $m = 2^{\binom{n}{2}}$, that is, it has $n + 2^{n^2-n+2} - 2$ vertices. We claim that T' is rigid, which suffices to prove the lemma. Let V denote the vertex set of T' (a union of U and the sets V_{ij}).

The sets V_{ij} are clearly all good, though possibly not maximal good. Hence, by Lemma 3.3, if $B \cap V_{ij} \neq \emptyset$ and B is maximal good, then $V_{ij} \subseteq B$.

The idea of the proof is as follows. First observe that automorphisms of the subtournament (V, \rightarrow) of T preserve the family of maximal good sets. We aim to show that by our construction of V , all non-singleton maximal good sets in V have different sizes, so in fact each is fixed setwise, and hence pointwise, by any automorphism. We then show that if some automorphism α of (V, \rightarrow) fixes pointwise all non-singleton maximal good subsets of V , then α fixes V pointwise.

Claim 2. If A is a good subset of V , then $|A \cap U| \leq 1$.

Proof of Claim 2. Suppose $u_1, u_2 \in A \cap U$, with $u_1 \neq u_2$. We have $u_1 \rightarrow V_{12} \rightarrow u_2$. Since A is good, we must have $V_{12} \subset A$: otherwise any $y \in V_{12} \setminus A$ separates u_1, u_2 , contradicting the fact that A is nice. Similarly, we have $u_2 \rightarrow V_{21} \rightarrow u_1$, and we must have $V_{21} \subset A$. But then we have $\{u_1, u_2\} \cup V_{12} \cup V_{21} \subseteq A$, and $u_1 \rightarrow V_{12} \rightarrow u_2 \rightarrow V_{21} \rightarrow u_1$. But then A is not totally ordered by \rightarrow , which contradicts the fact that A is good.

Thus, maximal good sets are unions of sets V_{ij} with at most one element of U added (this includes the case of a singleton point of U). Then by our choice of the sizes of the V_{ij} in the construction, any two non-singleton maximal good sets have different sizes. (For let the V_{ij} have sizes n_1, \dots, n_t , say. These were chosen as distinct powers of 2, and so all numbers of the form $n_{i_1} + \dots + n_{i_s}$ or $n_{i_1} + \dots + n_{i_s} + 1$ are

distinct.) Hence any automorphism of V fixes each non-singleton maximal good set setwise, and hence also pointwise since each is totally ordered and so rigid. Thus any automorphism fixes all elements of $V \setminus U$ pointwise, and so also fixes U pointwise; indeed, for each pair of elements of U there is some $Z \subset V \setminus U$ separating the pair, and so no automorphism can move points of U . \square

Corollary 3.4. *Let T be an infinite tournament with 2-homogeneous automorphism group. Then T is cofinally rigid.*

Proof. By Ramsey's Theorem, there is a subtournament of T of the form $\{x_i : i \in \mathbb{N}\}$ with $x_i \rightarrow x_j$ if and only if $i < j$ (or possibly with all arcs reversed). Clearly, if $i < j$, then $|\Gamma^+(x_j) \Delta \Gamma^+(x_i)| \geq j - i - 1$. By 2-homogeneity of G (and therefore $\text{Aut}(T)$), there is $d \in \mathbb{N} \cup \{\aleph_0\}$ such that if $x \neq y$ then $|\Gamma^+(x) \Delta \Gamma^+(y)| = d$. Hence, $d \geq n$ for each $n \in \mathbb{N}$, so $d = \aleph_0$.

We may suppose that T is not totally ordered by \rightarrow , since finite total orders are rigid. By Proposition 3.2, the proof of the corollary now reduces to the following claim.

Claim. For all distinct $x, y \in X$, the sets $\Gamma^+(x) \setminus \Gamma^+(y)$ and $\Gamma^+(y) \setminus \Gamma^+(x)$ are both infinite.

Proof of Claim. Suppose that for some $u, v \in X$ with $u \neq v$, the set $\Gamma^+(u) \setminus \Gamma^+(v)$ is infinite, but $\Gamma^+(v) \setminus \Gamma^+(u)$ is finite. Now, using 2-homogeneity, define an order relation $<$ on X , such that $x < y$ if and only if $\Gamma^+(x) \setminus \Gamma^+(y)$ is infinite. This is a G -invariant partial order on X , containing comparable pairs. By 2-homogeneity, it follows that $<$ is a total order, and it or its reverse agrees with \rightarrow . This contradicts the above assumption. \square

Proof of Proposition 3.1. As noted above, there is a G -invariant tournament T with vertex set X , whose arc set is a G -orbit on $X^{[2]}$. The proposition now follows immediately from 2-homogeneity and Corollary 3.4. \square

Proof of Proposition 1.1. This is immediate from Lemma 1.3 and Propositions 2.1 and 3.1. \square

4 Further remarks

The proof of Theorem 1.1 yields that there is a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for any $l \in \mathbb{N}$, if $H \leq G$ are closed permutation groups on an infinite set X with G primitive but not 2-transitive, and G and H have the same orbits on $X^{[n]}$ for all $n \leq f(l)$, then G and H have the same orbits on X^m for all $m \leq l$. An upper bound for f is given by the cardinality of V in terms of $|U|$ in the definition in the Introduction of a group G acting locally rigidly. By the proofs of Propositions 2.1 and 3.1, we obtain the following slight strengthening of Theorem 1.1, probably far from best possible. Observe that with m and k as in Remark 2.4, $\frac{1}{2}(2n + k(2m + k + 5)) \leq n + 2^{n^2 - n + 2} - 2$ for all $n > 1$, so the bound in the proof of Proposition 3.1 dominates.

Theorem 4.1. *Let G, H be closed permutation groups on the infinite set X , with G primitive but not 2-transitive on X , and with $H \leq G$. Let $n \in \mathbb{N}$, and suppose that G and H have the same orbits on the set $X^{[l]}$ for each $l \leq n + 2^{n^2-n+2} - 2$. Then G and H have the same orbits on X^m for each $m \leq n$.*

Theorem 1.1 requires the assumption of primitivity. For example, $\text{Aut}(\mathbb{Q}, <) \text{Wr} C_2$ is orbit equivalent to $\text{Sym}(\mathbb{Q}) \text{Wr} C_2$ (in the natural imprimitive action). However, a proof of Conjecture 1.2 should yield a lot of information about the imprimitive case.

A proof of Conjecture 1.2, at least if via local rigidity, would appear to require arguments considerably more involved than those of this paper. As an example, suppose that G is 2-primitive (that is, 2-transitive and with primitive point stabilisers) but not 3-homogeneous on the infinite set X . We conjecture that G acts locally rigidly. There is a G -invariant 3-hypergraph Γ on X , and we would like to show that Γ (possibly expanded by some other G -invariant relations) is locally rigid. Given $x \in X$, there is an induced graph Γ_x on $X \setminus \{x\}$ on which G_x acts primitively. However, it is not clear that local rigidity of Γ_x transfers to local rigidity of Γ , or that a straightforward induction on the degree of transitivity of G can be made to work. There may also be an approach to local rigidity of hypergraphs using [14, Lemma 2.5] and related results.

We cannot even prove the conjecture under the assumptions that X is countable and G is locally compact (that is, there is some finite $F \subset X$ such that all orbits of $G_{(F)}$ on X are finite). Even the case when G is countable is open. A first class to consider would be that of primitive groups with finite point stabiliser, for which Smith [22] gives a useful-looking version of the O’Nan-Scott Theorem.

However, as evidence for the conjecture, we observe that an obvious place to look for a counterexample, suggested by the family of closed supergroups of $\text{Aut}(\mathbb{Q}, <)$ listed in Conjecture 1.2, fails. Indeed, let $(T, <)$ be any of the countable 2-homogeneous trees (that is, semilinear orders) classified by Droste in [5]. There is a family of interesting primitive closed permutation groups associated with $\text{Aut}(T, <)$, namely the primitive Jordan permutation groups with primitive Jordan sets classified in [1]: we have in mind $\text{Aut}(T, <)$, the automorphism group of the ternary general betweenness relation on T induced from $<$, the automorphism group of the corresponding countable C -structure, a structure whose elements are a dense set of maximal chains in $(T, <)$, and the automorphism group of the corresponding D -relation (a quaternary relation on the set of ‘directions’ of the betweenness relation). It can be checked that each of these groups acts locally rigidly. We omit the details.

In [4] a permutation group G on X is defined to be *orbit-closed* if there is no $H \leq \text{Sym}(X)$ which properly contains G and is orbit-equivalent to G . Such G will be a closed permutation group, and Conjecture 1.2 asserts that if X is countably infinite then the only primitive closed permutation groups which are not orbit-closed are the proper subgroups of $\text{Sym}(X)$ listed in that conjecture. In [4] the authors define $G \leq \text{Sym}(X)$ to be a *relation group* if there is a collection R of finite subsets of X such that

$$G = \{g \in \text{Sym}(X) : \forall a \in \mathcal{P}(X)(a \in R \leftrightarrow a^g \in R)\}.$$

Clearly any relation group is orbit-closed. Also, by [4, Corollary 4.3], any *finite* primitive orbit-closed group is a relation group. We do not know whether this holds without finiteness, and in particular cannot answer the following question, to which Siemons drew our attention.

Question 4.2. *Is $\text{Aut}(\mathbb{Q}, <)$ the only primitive but not 2-transitive closed permutation group of countable degree which is not a relation group?*

As a small example, let Γ_3 be the universal homogeneous 2-edge-coloured graph with edges coloured randomly red or green; that is, the unique countably infinite homogeneous 2-edge-coloured graph such that for any three finite disjoint sets U, V, W of vertices, there is a vertex x not adjacent to any vertex in U , adjacent by a red edge to each element of V and by a green edge to each element of W . At first sight, $G = \text{Aut}(\Gamma_3)$ is not a relation group, but in fact it is a relation group; for we may take R to consist of the 2-sets joined by a red edge and the 3-sets which carry a green triangle.

Our remarks in the Introduction suggest a further question. Again, for convenience, we shall consider actions on a countably infinite set X . A subset Y of X is a *moiety* of X if $|Y| = |X \setminus Y|$.

Question 4.3. *Which primitive closed permutation groups G on a countably infinite set X have a regular orbit on moieties?*

To say that G has a regular orbit on moieties of X is the same as to say, in the language of [13], that any first order structure M on X with $G = \text{Aut}(M)$ has *distinguishing number 2*. Some results on this are obtained in [13]. For example, if M is a homogeneous structure such that the collection of finite structures which embed in it is a ‘free amalgamation class’, and $\text{Aut}(M)$ is primitive but for some k is not k -transitive, then $\text{Aut}(M)$ has a regular orbit on moieties. In particular, this holds for the random graph, as follows already from [10, Theorem 3.1]. On the other hand, as noted in [13] it is easily seen that $\text{Aut}(\mathbb{Q}, <)$ has no regular orbit on moieties; for if A is a moiety of \mathbb{Q} whose setwise stabiliser is trivial, then A is dense and codense in \mathbb{Q} , but the structure $(\mathbb{Q}, <, P)$, where P is a unary predicate naming a dense codense set, is homogeneous so admits 2^{\aleph_0} automorphisms.

This suggests the following strengthening of orbit-equivalence. Let us say that permutation groups G, H on the countably infinite set X are *strongly orbit-equivalent* if they have the same orbits on the power set $\mathcal{P}(X)$ of X (not just on *finite* subsets of X). The following conjecture is implied by Conjecture 1.2, for it is easily seen that the five closed groups containing $\text{Aut}(\mathbb{Q}, <)$ all have different orbits on $\mathcal{P}(\mathbb{Q})$. For example, $\text{Aut}(\mathbb{Q}, <)$ has an orbit consisting of increasing subsets of order type ω with rational supremum, but this family of sets is not invariant under the automorphism groups of the induced circular order or linear betweenness relation.

Conjecture 4.4. *Let G, H be strongly orbit-equivalent closed permutation groups on the countably infinite set X . Then $H = G$.*

Again, the assumption that the groups are closed is necessary. Stoller ([23], see also [16]) gives an example of a proper subgroup H of $G = \text{Sym}(\mathbb{N})$ which is strongly

orbit-equivalent to G ; namely, let H consist of those permutations g of \mathbb{N} such that there are two partitions, dependent on g , of \mathbb{N} into finitely many sets A_1, \dots, A_k and B_1, \dots, B_k (so $\mathbb{N} = A_1 \cup \dots \cup A_k = B_1 \cup \dots \cup B_k$, each partitions) such that for each $i = 1, \dots, k$, the element g induces an order isomorphism $(A_i, <) \rightarrow (B_i, <)$.

Finally, we mention a conjectural strengthening of Lemmas 1.4 and 2.3. It is a special case of a much stronger conjecture in [6].

Conjecture 4.5. *Let Γ be an infinite graph such that for any distinct vertices x, y the set $\Gamma(x)\Delta\Gamma(y)$ is infinite. Then Γ is locally rigid.*

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